Now, let $O, H, I,$ and N denote the circumcentre, the orthocentre, the incentre, and the centre of the nine-point circle, respectively. As usual, let R and s be the circumradius and the semi-perimeter of ΔABC. It is well-known that the incircle is internally tangent to the nine-point circle. It follows that (b) is equivalent to $OH \perp IN$. Since $\overrightarrow{NI} = \overrightarrow{OI} - \overrightarrow{ON} = \overrightarrow{OI} - \frac{1}{2}\overrightarrow{OH}$, the condition $OH \perp IN$ is equivalent to $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$. Using $2s\overrightarrow{OI} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}$ and $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, a simple computation shows that $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$ is itself equivalent to

$$s = -a\cos 2A - b\cos 2B - c\cos 2C \qquad (1)$$

(note that for example $\overrightarrow{OA} \cdot \overrightarrow{OB} = R^2 \cos 2C$, C being acute or not).

We successively rewrite (1) as

$$a(1 + 2\cos 2A) + b(1 + 2\cos 2B) + c(1 + 2\cos 2C) = 0$$
$$\sin A + 2\sin A\cos 2A + \sin B$$
$$+2\sin B\cos 2B + \sin C + 2\sin C\cos 2C = 0$$
$$\sin 3A + \sin 3B + \sin 3C = 0$$

(the latter because $2\sin A\cos 2A = \sin 3A - \sin A$, etc.).

As a result, (b) is equivalent to $\sin 3A + \sin 3B + \sin 3C = 0$, or, with the help of the familiar trig formulas, to $4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2} = 0$. Finally, (b) is equivalent to $90° = \frac{3A}{2}$ or $\frac{3B}{2}$ or $\frac{3C}{2}$ and (b) $\Longleftrightarrow$ (c).

7. Let a, b, c be nonnegative real numbers such that $a+b \le c+1$, $b+c \le a+1$ and $c+a \le b+1$. Prove that

$$a^2 + b^2 + c^2 \le 2abc + 1 .$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution by Alt.

First note that $a, b, c \le 1$. Indeed,

$$c + a \le b + 1 \Rightarrow c + 2a \le a + b + 1 \le c + 1 + 1 \Rightarrow 2a \le 2 \Rightarrow a \le 1\,,$$

and, similarly, $b, c \le 1$.

Let $x := 1 - a$, $y := 1 - b$, $z := 1 - c$. Then $x, y, z \in [0, 1]$, $a = 1 - x$, $b = 1 - y$, $c = 1 - z$

$$\begin{cases} a + b \le c + 1 \\ b + c \le a + 1 \\ c + a \le b + 1 \end{cases} \Longleftrightarrow \begin{cases} z \le x + y \\ x \le y + z \\ y \le z + x \end{cases} \Longleftrightarrow |x - y| \le z \le x + y,$$

and the original inequality becomes

$$\begin{aligned}
&(1-x)^2+(1-y)^2+(1-z)^2 \le 2(1-x)(1-y)(1-z)+1\\
&\iff\ x^2+y^2+z^2 \le 2(xy+yz+zx)-2xyz\\
&\iff x^2+y^2-2xy \le 2z(x+y-xy)-z^2\\
&\iff (x-y)^2 \le 2z(x+y-xy)-z^2\,. \qquad (1)
\end{aligned}$$

Let $f(z) = 2z(x+y-xy) - z^2$. Since $z \in [|x-y|, x+y]$ then $\min\limits_z f(z) = \min\{f(|x-y|), f(x+y)\}$, and, therefore,

$$\begin{aligned}
(1) &\iff (x-y)^2 \le \min_z \left(2z(x+y-xy)-z^2\right)\\
&\iff (x-y)^2 \le \min\{f(|x-y|), f(x+y)\} \qquad (2)\\
&\iff \begin{cases} (x-y)^2 \le f(|x-y|)\\ (x-y)^2 \le f(x+y)\end{cases}\\
&\iff \begin{cases} (x-y)^2 \le 2|x-y|(x+y-xy)-(x-y)^2\\ (x-y)^2 \le 2(x+y)(x+y-xy)-(x-y)^2\end{cases}\\
&\iff \begin{cases} (x-y)^2 \le |x-y|(x+y-xy)\\ x^2+y^2 \le (x+y)(x+y-xy)\end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
x^2+y^2 \le (x+y)(x+y-xy) &\iff (x+y)xy \le 2xy\\
&\iff 0 \le xy(2-x-y)
\end{aligned}$$

and

$$(x-y)^2 \le |x-y|(x+y-xy) \iff 0 \le |x-y|(x+y-xy-|x-y|)\ .$$

Since $x, y \in [0,1]$ then the inequality $0 \le xy(2-x-y)$ obviously holds and

$$\begin{aligned}
|x-y| \le x+y-xy &\iff xy-x-y \le x-y \le x+y-xy\\
&\iff \begin{cases} xy-x \le x\\ -y \le y-xy\end{cases} \iff \begin{cases} 0 \le x(2-y)\\ 0 \le y(2-x)\end{cases}\ .
\end{aligned}$$